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Coherent perfect rotation

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Two classes of conservative, linear, optical rotary effects (optical activity and Faraday rotation) are distinguished by their behavior under time reversal. Faraday rotation, but not optical activity, is capable of coherent perfect rotation, by which we mean the complete transfer of counterpropagating coherent light fields into their orthogonal polarization. Unlike coherent perfect absorption, however, this process is explicitly energy conserving and reversible. Our study highlights the necessity of time-reversal-odd processes (not just absorption) and coherence in perfect mode conversion and thus informs the optimization of active multiport optical devices.

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We use the term coherent perfect rotation (CPR) to denote the linear, conservative, reversible transfer of *any* fixed input polarization state of coherent counterpropagating light fields completely into its orthogonal polarization. There are (many) single-port methods for completely rotating the light into its orthogonal polarization state, but their activity does not depend on, and so cannot be controlled by, the phase of the light field as in an intrinsically two-port process such as CPR. Perhaps somewhat surprisingly, and of utility for general applications involving any type of coherent perfect mode conversion, below we explicitly show that CPR is possible only if the underlying conversion process has the correct fundamental symmetry, namely, it (or, if of mixed symmetry, some part of it) must be odd under time reversal.

The name "coherent perfect rotation" comes from the significant formal phenomenological parallels of this process to coherent perfect absorption (CPA) [1,2], the so-called "antilaser" state that has attracted significant interest recently. CPA in one spatial dimension refers to the (reflectionless and thus complete) absorption of phased counterpropagating waves incident on a medium. As a linear process, CPA is possible only as a nonconservative process, and is modeled in Refs. [1,2] using a non-Hermitian Hamiltonian. In its original formulation, this non-Hermitian Hamiltonian included either absorption or gain and thus explicitly breaks the time-reversal invariance of the underlying fundamental processes. Recall that CPA is strikingly distinct from critical coupling, in which the absorption and coupling in a device are tuned to cause reflectionless absorption. Although both are linear, in the case of critical coupling there may be multiple ports but the reflectionless process is incoherent in each port, that is, it is not a coherent process depending on the phase at each port. In analogy with critical coupling, CPA occurs only for isolated values of the absorption index and the optical length. CPA phenomena have been illustratively generalized in PT-invariant systems [3,4], leading to a fertile way to explore many subtleties in optical processes [5,6].

CPR differs fundamentally from CPA in that it is a conservative, reversible process that can be understood simply in terms of an explicitly Hermitian Hamiltonian. One consequence is that the CPR resonances often appear as spectroscopic doublets having the same spatial symmetry. In the analogous system with CPA, the resonances appear as singlets. There are, however, many important phenomenological correspondences between CPR and CPA and likely any coherent perfect process. These include the necessity of an underlying T-odd process, the occurence of the phenomenon at isolated values of control parameters representing the strength of that process, and the fact that these isolated values are significantly below those for which the associated system is critically coupled. After developing the theory of CPR and its consequences we propose an experimental demonstration of CPR and briefly discuss applications it may afford.

We adopt a 4×4 transfer matrix approach to describe linear optical transport of a monochromatic ray moving back and forth along the $\hat{z}$ axis,

$$\mathcal{M} = \begin{pmatrix} M & C \\ B & M' \end{pmatrix} \qquad \text{with } \vec{v}_{i+1} = \mathcal{M}_i \vec{v}_i, \tag{1}$$

where the M's, B, and C are 2×2 (in general complex) matrices; here we are working in the basis where the local field (complex) amplitudes are $\vec{v} = (E_x, H_y, E_y, -H_x)$. The more familiar single-polarization form of the transport is in terms of the 2×2 M matrix (take $B = C = 0$). We work in units in which the familiar propagation eigenstates of a single polarization in the vacuum are $\vec{e}_R = (E_x, H_y) = (1,1)$ for a right-moving wave and $\vec{e}_L = (-1,1)$ for a left-moving wave. Thus, for review, we represent the coherent scattering from a linear material whose (2×2) transfer matrix is M by $\vec{e}_{\text{in}} = (1,1) + r(-1,1)$ as the incident fields from the left and $\vec{e}_{\text{out}} = t(1,1) = M\vec{e}_{\text{in}}$ for the fields on the right, with r and t denoting the reflection and transmission amplitudes (generally complex numbers). For reference, solving the transport in this basis gives $t = 2(m_{11}m_{22} - m_{12}m_{21})/(m_{11} + m_{22} - m_{12} - m_{21})$ and $r = (m_{11} - m_{22} + m_{12} - m_{21})/(m_{11} + m_{22} - m_{12} - m_{21})$, where the m_{ij} are the matrix elements of M (note the difference in basis from Ref. [7]).

In the 2×2 case, T symmetry indicates that real diagonal elements of M are T even whereas real off-diagonal elements are T odd. In general, matrices C and B in $\mathcal{M}$ can each be written as a sum of T-even and T-odd parts. Thus in the chosen basis the T-even part is of the form

$$\{C \text{ or } B\}_{T\,\text{even}} = \begin{bmatrix} \text{Re} & \text{Im} \\ \text{Im} & \text{Re} \end{bmatrix}, \tag{2}$$

where Im (Re) stands for imaginary (real) matrix elements. Note that these elements can all be different from one another.

In contrast, for the T-odd part of the B and C matrices,

$$\{C \text{ or } B\}_{T\text{ odd}} = \begin{bmatrix} \text{Im} & \text{Re} \\ \text{Re} & \text{Im} \end{bmatrix}. \tag{3}$$

where, again, all entries could be different. Although T-odd pieces in the 2×2 M are associated with absorption or gain there are combinations of these T-odd matrix elements in C that conserve the total power of the 4×4 system. Materials without linear birefringence are $O(2)$ symmetric about the axial direction, implying $M = M'$ and $B = -C$, regardless of the T symmetry of the matrix elements.

In steady state, the local power flux will be constant for a conservative system. A local expression for the power flux in the chosen basis is $\sim\vec{v}^{\,\dagger}\mathcal{P}\vec{v}$ where $\mathcal{P} = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$ in which for each polarization $P = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The statement that the transport is conservative is thus $\mathcal{M}^\dagger\mathcal{P}\mathcal{M} = \mathcal{P}$. This leads to important constraints on the matrix elements of M and B,C. In the more familiar 2×2 formulation of transport for a single polarization, a conservative system satisfies $M^\dagger P M = P$, indicating that m_{11} and m_{22} must be purely real, while m_{12} and m_{21} must be purely imaginary [$\det(M) = 1$ is automatic in one-dimensional (1D) linear transport as it preserves the E_x, H_y commutator], and is T even. For example, for normal incidence on a purely dielectric material of thickness L and index n,

$$M = \begin{bmatrix} \cos\delta & \frac{i}{n}\sin\delta \\ in\sin\delta & \cos\delta \end{bmatrix}, \tag{4}$$

where $\delta = nk_0L$ and k_0 is the vacuum wave number. Similarly, in the full 4×4 transport, T symmetry and power conservation are not identical. If we restrict consideration to T-even, conservative transport in the uniaxial case ($M = M'$ and $B = -C$) then power conservation gives

$$m_{12}c_{21} - m_{21}c_{12} = m_{22}c_{11} - m_{11}c_{22} \tag{5}$$

along with $c_{21}/c_{12} = m_{21}/m_{12}$ and $c_{11}/c_{22} = m_{11}/m_{22}$. Combining these equations indicates that $c_{ij} = \chi m_{ij}$ with χ a real constant. Then power conservation indicates $(1+\chi^2)M^\dagger P M = P$ for transport with rotation in a simple dielectric material for which

$$M = \cos\gamma\begin{bmatrix} \cos\delta & \frac{i}{n}\sin\delta \\ in\sin\delta & \cos\delta \end{bmatrix}$$

and $\chi = \tan\gamma$. Thus only a single parameter, γ, governs the overall rotation in the T-even case, as would be the case for optical activity in which γ is proportional to the product of the concentration of chiral centers and sample length.

Assuming both that the components of M remain T even and that the system is uniaxial, the case of conservative, T-odd C power conservation reduces to

$$\det M - \det C = 1 \tag{6}$$

and

$$m_{11}c_{22} + m_{22}c_{11} = m_{21}c_{12} + m_{12}c_{21}. \tag{7}$$

Thus, studying the conjugation and scaling symmetry of the above equations, we see that there are three (real) parameters that determine the longitudinal T-odd polarization mixing. One of these parameters is the ordinary Faraday rotation parameter (the Verdet constant times the applied longitudinal magnetic field). The other two parameters in a general solution of Eqs. (6) and (7) are less familiar although they lead to the same phenomena.

The adjective "coherent" in CPA and CPR indicates their reliance on the relative phase between the counterpropagating light fields in achieving perfect mode conversion. Thus CPA and CPR are necessarily two-port processes, in contrast to critical coupling [8–10], itself sometimes referred to as one-port CPA. Having established the formalism and symmetry, we now show that CPR is possible only using T-odd processes such as Faraday rotation.

As noted in the original formulation [1], CPA can be understood via 2×2 transfer matrices. In CPA there are incoming fields only, and in our choice of basis, these are $\vec{v}_l = (E_x, H_y) = (1,1)$ and $\vec{v}_r = f(-1,1)$ (note that f is complex). These fields are related via the transfer matrix as $\vec{v}_r = M\vec{v}_l$, which in terms of the matrix elements of M indicates that CPA requires the condition $m_{11} + m_{22} + m_{12} + m_{21} = 0$. In terms of a fixed optical element size, this (complex) equation yields both the wavelength of the CPA pole in the S matrix and the critical value of the dissipative coupling (which necessarily has T-odd components in M).

It is straightforward to find the location of a CPR resonance using the 4×4 basis. For fields on the left, take $\vec{v}_l = (1,1,-l,l)$ where l is the amplitude of the outgoing rotated wave. On the right, take $\vec{v}_r = (-d,d,s,s)$; this configuration thus consists of only incoming fields of one polarization and outgoing fields of the orthogonal polarization only, the CPR state. In analogy with the CPA state, these boundary conditions lead to a condition on the size, wavelength, and rotary power of the system. For uniaxial systems with the 4×4 form of $\mathcal{M}$ as described earlier, we require

$$M\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C\begin{pmatrix} -1 \\ 1 \end{pmatrix}l = \begin{pmatrix} -1 \\ 1 \end{pmatrix}d \tag{8}$$

and

$$-C\begin{pmatrix} 1 \\ 1 \end{pmatrix} + M\begin{pmatrix} -1 \\ 1 \end{pmatrix}l = \begin{pmatrix} 1 \\ 1 \end{pmatrix}s. \tag{9}$$

No optically active, uniaxial, conservative process ever solves the above pair, and thus it cannot be used to achieve CPR. For this case, as indicated in the preliminaries, $C \sim M$ and thus $\mathcal{M} = \left[\begin{smallmatrix} M\cos\gamma & -M\sin\gamma \\ M\sin\gamma & M\cos\gamma \end{smallmatrix}\right]$, for γ proportional to the concentration-length product of the chiral centers. Using this form in Eqs. (8) and (9) and eliminating l, s, and d, we arrive at the single constraint

$$-(m_{11} - m_{22})^2 + (m_{12} - m_{21})^2 = 4\cos^2\gamma. \tag{10}$$

The power conservation discussed earlier indicates that m_{11} and m_{22} must be purely real in this basis, such that m_{12} and m_{21} are purely imaginary; thus Eq. (10) can never be achieved unless both sides are identically zero. If so, then both $m_{11} = m_{22}$ and $m_{12} = m_{21}$. Thus the condition $\det(M) = 1$ would imply that there exists some angle ϕ such that $m_{11} = \cos\phi = m_{22}$ and $m_{12} = i\sin\phi = m_{21}$. For $\phi \neq 0$, this case would correspond to a material that has a net index of refraction of unity. Alternatively, plugging the choice $\phi = 0$ into Eqs. (8) and (9), the equations become

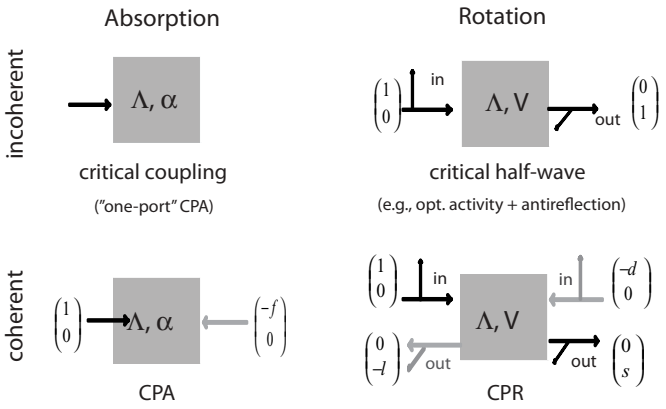

FIG. 1. CPA and CPR are distinct from each other and the associated critical coupling and critical rotation [8–10]. For a fixed value of Λ, the system's length in terms of the vacuum wavelength, critical coupling and CPA occur at a particular value of the absorption index α. Similarly, rotation is parametrized by the product of the material's Verdet and the magnetic-field, V. Only CPA and CPR depend upon both the amplitude and relative phase of the counterpropagating beams. Here, for simplicity, the ordered pairs indicate orthogonal polarizations and the sign indicates the direction of propagation.

degenerate, relaxing the requirement on the index although yielding a solution for any inputs (1 or d in any relation, since $M = \mathbf{1}$) independently. This is not CPR; it is instead the rotational analog of critical coupling (Fig. 1). To reiterate, such a system conservatively rotates the polarization of light from any given polarization state completely into the orthogonal state whether it is illuminated from one side or the other, independent of any phase relationship between the incoming fields. Indeed, a single slab of an optically active material can be tuned in width and chiral concentration to create this analog of critical coupling for rotation. There are likely to be other ways to achieve this rotational analog of critical coupling, including one we discuss below, but again, this is not CPR.

The main idea of this Rapid Communication is that CPR is achievable with T-odd rotation, as we now show analytically for a slab dielectric Faraday rotator. The M and C in the chosen basis for a slab are [11]

$$M = \frac{1}{2}\begin{bmatrix} C_1 + C_2 & i(S_1/n_1 + S_2/n_2) \\ i(n_1 S_1 + n_2 S_2) & C_1 + C_2 \end{bmatrix} \quad (11)$$

and

$$C = \frac{1}{2}\begin{bmatrix} i(C_1 - C_2) & -(S_1/n_1 - S_2/n_2) \\ -(n_1 S_1 - n_2 S_2) & i(C_1 - C_2) \end{bmatrix}, \quad (12)$$

where $C_{1,2}$ ($S_{1,2}$) refer to the cosine (sine) of $\delta_{1,2} = n_{1,2} k_0 L$ in which n_1,n_2 are the indices of refraction of the left- and right-circular polarization in the slab, k_0 refers to the vacuum wave vector, and L is the thickness of the slab. For a dielectric slab in an external magnetic field pointing along the direction of propagation, $\delta n = n_1 - n_2$ is proportional to the product of the Verdet constant and the magnetic field. Note that this C given by Eq. (12) has the requisite symmetry of Eq. (3) and is conservative.

The system Eqs. (8) and (9) gives four (complex) relations for three complex quantities (d,s,l), so, being overdetermined,

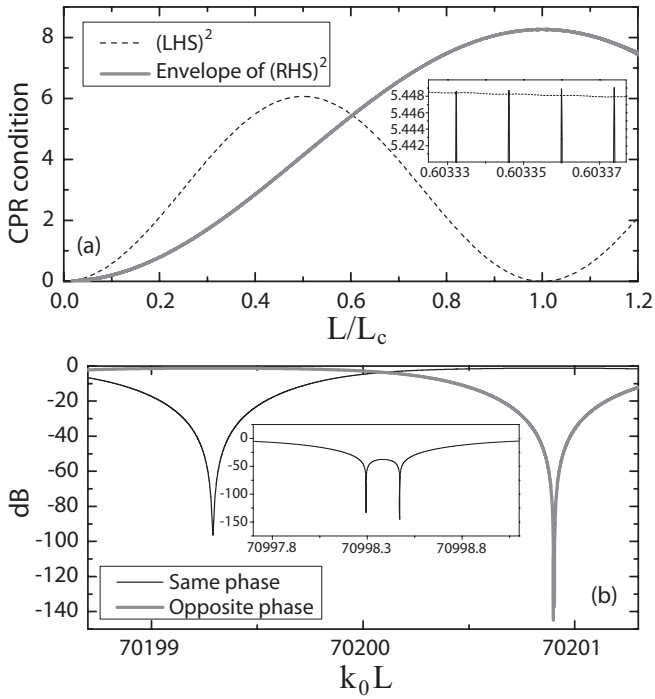

FIG. 2. (a) The LHS (black dashed line) and the envelope of the fast-oscillating RHS (thick gray line) of the CPR condition Eq. (13) plotted as a function of L/L_c. The inset is a small portion of the full graph near the first occurence of the CPR resonance and where the fast oscillations of the RHS are also shown. (b) Total reflected intensity in the same polarization as the input fields as a function of the length L multiplied by the vacuum wave number k_0. The thin line corresponds to the case where the counterpropagating fields have the same relative phase and the thick line to the case where they are 180° out of phase. The inset shows a CPR "doublet" at larger $k_0 L$ at which the line splitting is more prominent.

demands a condition on $n_{1,2}$ and $k_0 L$ which may or may not be physically satisfiable. Algebra shows this condition to be

$$\left(n_1 + \frac{1}{n_1}\right) S_1 C_2 - \left(n_2 + \frac{1}{n_2}\right) S_2 C_1 = \pm\left[\left(n_1 - \frac{1}{n_1}\right) S_1 - \left(n_2 - \frac{1}{n_2}\right) S_2\right]. \quad (13)$$

Whenever this condition is satisfied, the fields fall into the parity ($p = \pm 1$) eigenstates $l = ps$ and $d = p$. Again, these are necessarily two-port resonances, as is CPA, and thus examples of CPR states. A numerical solution is shown in Fig. 2 for terbium gallium garnet with $\bar{n} = (n_1 + n_2)/2 = 1.95$ subject to a coherent 632.8 nm light source and $\delta n = 2.7 \times 10^{-5}$ produced by a 1 T external field. In Fig. 2(a) the square of the left-hand side (LHS) of Eq. (13) is plotted as a dashed line and the (envelope of the) square of the right-hand side (RHS) as a gray one. The first of many CPR states exists under these conditions at $L/L_c \approx 0.603$, where L is the length of the slab and L_c is the critical half-wave rotation length. It is rather easy to understand some general trends in the location of the CPR resonances in λ. Increasing $\bar{n} = (n_1 + n_2)/2$ or δn brings the location of the first CPR resonance to lower $k_0 L$, as would be the case in CPA with δn playing the role of α, the absorption constant. Thus for a fixed L and a given range

of k_0, there is a threshold δn at which CPR states first appear, again reminiscent of CPA.

Also in Fig. 2(b) is a graph of the total power reflected with the same polarization as the input fields for this case, clearly indicating the first CPR resonance near $k_0 L \approx 70\,201$, or $L \approx 7.07$ mm. Notably, for this simple slab geometry, all resonances come in pairs of the same parity and are part of a parity-alternating series of pairs of resonances. As in CPA, these CPR resonances are bound-state-like (zero width). Unlike CPA where there is but one resonance, for CPR, given $\bar{n}$, L, and a range of k_0, there are many, and they occur generically in "doublets" with the same spatial symmetry.

Finally, just as one can reach critical coupling in a one-port version of CPA, one can see that for particular values of n_1, n_2, and $k_0 L$ there can be a degeneracy of the positive and negative parity resonances. For Fig. 2, this occurs for $k_0 L \approx 116\,355$. At degeneracy, taking linear combinations of the CPR resonances yields the (incoherent) critical rotator solutions (in detail they are at $S_1 = 0 = S_2$ and $C_1 = -C_2 = \pm 1$). These are optically indistinguishable from the critical rotator described earlier that relied on optical activity.

An experimental verification of CPR is planned using a high-Verdet-constant glass. The CPR resonances are thin, indicating that small changes in a substantial magnetic field (or in the material itself) may be readily detectable through changes in the extinction of a reflected polarization. At the level of technological application, note that an optical modulator based on CPA will necessarily have limited dynamic range as the material will always absorb some of the light even when not in CPA. A CPR-based optical modulator may not suffer the same limitation. This work has also stimulated theoretical investigations of more general coherent perfect multichannel conversion processes in nonlinear optics.

In conclusion, we have shown that Faraday rotation has the appropriate symmetry to manifest coherent perfect rotation and have analytically developed an example of CPR in a dielectric Faraday slab rotator. CPR and CPA have important phenomenological correspondences, but CPR is conservative and reversible so can be described in terms of a finite-dimensional manifestly Hermitian Hamiltonian. In light of the findings above it is likely that any coherent perfect conversion process requires the fundamental symmetry of the underlying process to be T odd.

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